Complex Plane: Polar Form and Integer Powers

Polar Form, Rotation, De Moivre's Formula, and Integer Powers

Lecture Notes

September 10, 2026

1 Polar Form of a Complex Number

A complex number z in polar form is given by its absolute value $|z|$ and an angle θ: in real coordinates, $(r\cos\theta, r\sin\theta)$; in complex notation, $z = |z|(\cos\theta + i\sin\theta)$.

Using the (temporary) notation $\operatorname{cis}\theta := \cos\theta + i\sin\theta$: $|\operatorname{cis}\theta| = 1$.

The argument

The angle $\theta = \arg z$ is not unique: $\arg z = \theta + 2k\pi$ for any $k \in \mathbb{Z}$. The **principal value** is restricted to $-\pi < \operatorname{Arg} z \le \pi$. Also $\arg \bar{z} = -\arg z = \{\theta : -\theta \in \arg z\}$. While the set $2\arg z$ is well defined independent of the chosen representative, $\dfrac{\arg z}{2}$ does depend on it.

Why the asymmetry? Fix one representative θ with $z = |z|\operatorname{cis}\theta$; every other legal choice is $\theta + 2k\pi$ for some $k \in \mathbb{Z}$, since two angles describe the same point of the plane exactly when they differ by an integer multiple of 2π. Doubling,

$$2(\theta + 2k\pi) = 2\theta + (2k)\cdot 2\pi,$$

which differs from 2θ by an integer multiple of 2π *for every* k. So $\operatorname{cis}(2\theta)$ comes out the same no matter which representative θ we started from: $2\arg z$ is well defined as a point on the unit circle, i.e. as a residue class mod 2π.

Halving is different:

$$\frac{\theta + 2k\pi}{2} = \frac{\theta}{2} + k\pi,$$

and $k\pi$ is an integer multiple of 2π only when k is *even*. When k is odd, the representative shifts by π, so

$$\operatorname{cis}\left(\frac{\theta}{2} + k\pi\right) = -\operatorname{cis}\left(\frac{\theta}{2}\right) \qquad (k \text{ odd}),$$

a genuinely different complex number. The representatives of $\arg z$ split into two families – one giving $\operatorname{cis}(\theta/2)$, the other $-\operatorname{cis}(\theta/2)$ – so $\dfrac{\arg z}{2}$ is only well defined mod π, not mod 2π: it does not pick out a single point of the unit circle, but a genuine two-element ambiguity. (This is exactly why $\sqrt{z}$ has two values, differing by a sign, rather than one.)

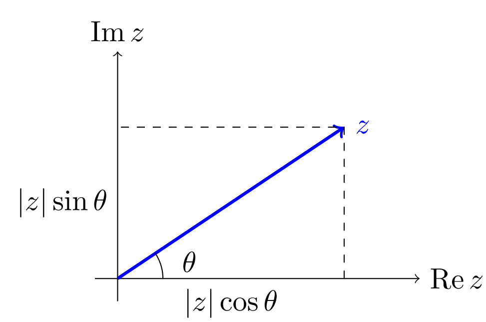

Figure 1: Polar form: z as a vector of length $|z|$ at angle $\theta = \arg z$ from the positive real axis, with real and imaginary parts $|z|\cos\theta$ and $|z|\sin\theta$.

2 Rotation as Multiplication

Multiplication of complex numbers geometrically represents rotation and dilation. Rotation of $\mathbb{R}^2$ is the linear map with matrix $M_{\operatorname{cis}\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

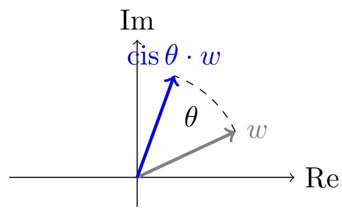

Figure 2: Multiplication by $\operatorname{cis}\theta$ rotates a vector w by angle θ about the origin, preserving its length.

Multiplying by $\operatorname{cis}\theta$ gives $\operatorname{cis}\theta \cdot w = M_{\operatorname{cis}\theta}\vec{w}$, rotating w by angle θ, so $\arg(\operatorname{cis}\theta \cdot w) = \arg w + \theta$. Since $z = |z| \operatorname{cis}(\arg z)$, the product zw is w rotated by $\arg z$ and dilated by $|z|$.

Theorem (arguments and moduli)

Multiplication: $\arg(zw) = \arg z + \arg w$, $|zw| = |z||w|$, where $\arg z + \arg w$ denotes the sum of the two residue classes modulo 2π (i.e. $\{\theta_1 + \theta_2 : \theta_1 \in \arg z,\ \theta_2 \in \arg w\}$, itself a single residue class mod 2π).

Division: $\arg(z/w) = \arg z - \arg w$, $|z/w| = |z|/|w|$ (for $w \neq 0$).

Proof. For multiplication: take $\theta_1 \in \arg z$, $\theta_2 \in \arg w$, and add. For division: $\arg(1/w) = \arg(\bar{w}/|w|^2) = -\arg w$; apply this to multiplication. $\square$

Remark. The identity $\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w$ does *not* always hold for the *principal* argument. For example, if $z = w = -i$, then $\operatorname{Arg} z = \operatorname{Arg} w = -\pi/2$, but $zw = -1$ and $\operatorname{Arg}(-1) = \pi \neq -\pi/2 + (-\pi/2)$.

3 Trigonometry and De Moivre's Formula

Trigonometry simplifies via $\operatorname{cis}(\theta_1 + \theta_2) = \operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2$: expanding,

$$\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2),$$

and equating real/imaginary parts recovers the standard sum formulas

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \qquad \sin(\theta_1 + \theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2.$$

De Moivre's Formula

$$\cos(n\theta) = \frac{1}{2}\big((\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\big) = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k \binom{n}{2k} \sin^{2k}\theta\cos^{n-2k}\theta,$$

$$\sin(n\theta) = -\frac{i}{2}\big((\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n\big) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \binom{n}{2k+1} \sin^{2k+1}\theta\cos^{n-1-2k}\theta.$$

4 Powers and Roots

Integer Powers

$z^n = |z|^n \operatorname{cis}(n\arg z)$ for $n \in \mathbb{N}$ or $n \in \mathbb{Z}$.

Example 1. $i^{239} = 1\cdot\operatorname{cis}(239\cdot\pi/2) = \operatorname{cis}(3\pi/2) = -i$.

Example 2. $(1+i)^{239} = (\sqrt{2})^{239}\operatorname{cis}(239\cdot\pi/4) = \sqrt{2}\cdot 2^{119}\cdot\operatorname{cis}(7\pi/4) = 2^{119}(1-i)$.